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Heat Conduction in a Random Medium

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The method of time-ordered cumulants is used to investigate the behavior of heat pulses in a one-dimensional medium in which the thermal conductivity is random. A partial differential equation is obtained for the average temperature profile; it is the heat equation modified by the addition of a fourth-order spatial derivative. A solution is obtained by asymptotic series. The first two spatial moments of the average temperature profile are evaluated and are shown to tend to those of a Gaussian when t is large. Finally, an equation is obtained for the covariance function.

KEY WORDS: Heat conduction; random medium; Gaussian distributions.

1. INTRODUCTION

It is our purpose in this paper to investigate heat conductivity in a random medium. Part of our interest is the physical phenomenon itself, while part is the mathematical technique used. The method of ordered cumulants is used to render the problem in an exactly tractable form, which simplifies even further as a result of our assumption that the time correlation involved is a delta function. This method is in dramatic contrast with other methods, which necessarily involve approximation procedures even though they treat precisely the same problem.

A very closely related procedure can be applied to the phenomenon of wave propagation in a random medium, and the consequence is that, on the average, the wave propagation becomes damped. Here, however, we begin with a dissipative process, heat conduction, and the stochastic nature of the

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conductivity produces, on the average, an antidamping behavior. A Fourier integral representation, consequently, is not valid, and we present an asymptotic series solution as well as an explicit calculation of the first few moments of the solution. It is observed that asymptotically in time the heat conduction behaves more and more just like it would behave in the absence of stochasticity. This result, coupled with the antidamping nature of the stochastically induced conduction, remains an intriguing dilemma.

We close our analysis with a presentation of a technique for the derivation of the Fokker–Planck equation which is associated with stochastic heat conduction. This technique undoubtedly has applicability in many other stochastic problems involving fields, such as is the case with heat conduction and wave propagation.

2. DIFFERENTIAL EQUATION SATISFIED BY MEAN SOLUTION

The equation under investigation is

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\nu(x, t)\frac{\partial T}{\partial x}\right] \tag{1}$$

where $T(x, t)$ is the temperature and $\nu(x, t)$ is the thermal conductivity (taken to be a stochastic function of space and time). If we introduce the spatially integrated temperature profile

$$S(x, t) = \int_{-\infty}^{x} T(x', t)\, dx' \tag{2}$$

then it can be shown that Eq. (1) reduces to

$$\partial S/\partial t = \nu(x, t)\, \partial^2 S/\partial x^2 \tag{3}$$

The stochastic thermal conductivity is taken to be of the form

$$\nu(x, t) = \nu_0[1 + \tilde{\mu}(x, t)] \tag{4}$$

where

$$\langle\tilde{\mu}(x, t)\rangle = 0, \qquad \langle\nu(x, t)\rangle = \nu_0 > 0 \tag{5}$$

and

$$\langle\tilde{\mu}(x, t)\tilde{\mu}(x', t')\rangle = 2R(|x - x'|)\, \delta(t - t') \tag{6}$$

For definiteness, it is further stipulated that $\tilde{\mu}$ is a Gaussian process.

The formal solution to the stochastic equation (3) can be expressed in terms of the time-ordered exponential[1]

$$S(x, t) = \underset{\leftarrow}{T}\left\{\exp\left[\int_0^t \nu(x, s)\frac{\partial^2}{\partial x^2}\, ds\right]\right\}S(x, 0) \tag{7}$$

The time-ordered exponential is defined by

$$\underset{\leftarrow}{T}\exp\int_0^t f(s)\,ds \equiv 1 + \int_0^t f(s)\,ds + \int_0^t\int_0^{t_1} f(t_1)f(t_2)\,dt_1\,dt_2 + \sum_{n=3}^{\infty}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} f(t_1)\cdots f(t_n)\,dt_1\cdots dt_n \tag{8}$$

for any function or operator $f(t)$. When the commutator of $f(t)$ at two different times vanishes, i.e.,

$$[f(t), f(s)] = 0 \tag{9}$$

then the time-ordered exponential reduces to the ordinary exponential. In Eq. (8),

$$f(t) \equiv \nu(x, t)\,\partial^2/\partial x^2 \tag{10}$$

and

$$[\nu(x, t)\,\partial^2/\partial x^2, \nu(x, s)\,\partial^2/\partial x^2] \neq 0 \tag{11}$$

Equations (3) and (4) provide an example of a multiplicative stochastic process.[2]

The solution given by Eq. (7) can be averaged using time-ordered cumulants[1]; the final result is

$$\langle S(x, t)\rangle = \underset{\leftarrow}{T}\exp\left\{\int_0^t [g^{(1)}(s) + g^{(2)}(s)]\,ds\right\}S(x, 0) \tag{12}$$

where

$$\int_0^t g^{(1)}(s)\,ds = \int_0^t \langle\nu(x, s)\rangle\frac{\partial^2}{\partial x^2}\,ds = t\nu_0\frac{\partial^2}{\partial x^2} \tag{13}$$

$$\begin{aligned}\int_0^t g^{(2)}(s)\,ds &= \int_0^t\int_0^{t_1} \nu_0^2\langle\tilde{\mu}(x, t_1)\frac{\partial^2}{\partial x^2}\tilde{\mu}(x, t_2)\rangle\frac{\partial^2}{\partial x^2}\,dt_2\,dt_1\\ &= t\nu_0^2\frac{\partial^2}{\partial y^2}R(|x - y|)_{y=x}\frac{\partial^2}{\partial x^2}\\ &= t\nu_0^2\left[\frac{\partial^2}{\partial y^2}R(|x - y|)\bigg|_{y=x}\right]\frac{\partial^2}{\partial x^2} + t\nu_0^2R(0)\frac{\partial^4}{\partial x^4}\end{aligned} \tag{14}$$

Only the first two cumulants are needed here because the second moment of $\tilde{\mu}(x, t)$, as given by Eq. (6), contains a Dirac delta function in the time variable. Employing the abbreviation

$$R''(0) \equiv \frac{\partial^2}{\partial y^2}R(|x - y|)_{y=x} \tag{15}$$

we can write Eq. (12) in the form

$$\langle S(x, t)\rangle = \exp\left\{t\nu_0\frac{\partial^2}{\partial x^2} + t\nu_0^2\left[R''(0)\frac{\partial^2}{\partial x^2} + R(0)\frac{\partial^4}{\partial x^4}\right]\right\}S(x, 0) \tag{16}$$

Therefore, $\langle S(x, t)\rangle$ satisfies the partial differential equation

$$\frac{\partial}{\partial t}\langle S(x, t)\rangle = \left\{[\nu_0 + \nu_0{}^2R''(0)]\frac{\partial^2}{\partial x^2} + \nu_0{}^2R(0)\frac{\partial^4}{\partial x^4}\right\}\langle S(x, t)\rangle \tag{17}$$

Upon using Eq. (2) and taking the x derivative of Eq. (17), we obtain

$$\frac{\partial}{\partial t}\langle T(x, t)\rangle = \left\{[\nu_0 + \nu_0{}^2R''(0)]\frac{\partial^2}{\partial x^2} + \nu_0{}^2R(0)\frac{\partial^4}{\partial x^4}\right\}\langle T(x, t)\rangle \tag{18}$$

which is the equation that the average temperature profile $\langle T(x, t)\rangle$ satisfies. Note the existence of the fourth-order spatial derivative on the right-hand side, and the modification of ν_0 by $\nu_0 R''(0)$ in the coefficient of the $\partial^2/\partial x^2$ term.

On physical grounds, it is natural to expect that $R(0) > 0$. Consequently, the $\partial^4/\partial x^4$ term will cause the coefficient of a spatial Fourier integral representation for $\langle T(x, t)\rangle$ to blow up. Nevertheless, the form of Eq. (18) makes it clear that the complete x integral of $\langle T(x, t)\rangle$, which is $\langle S(\infty, t)\rangle$ according to Eq. (2), is constant in time, provided $\langle T(x, t)\rangle$ vanishes at $x = \pm\infty$. This combination of circumstances forces us to examine an asymptotic series representation for $\langle T(x, t)\rangle$ instead of a Fourier integral representation.

3. SOLUTION BY ASYMPTOTIC SERIES

Let us take as the initial condition

$$T(x, 0) = T_0\,\delta(x - x_0) \tag{19}$$

With the abbreviations

$$c_1 \equiv \nu_0 + \nu_0{}^2R''(0), \qquad c_2 \equiv \nu_0{}^2R(0) \tag{20}$$

the solution to Eq. (18) may be written as

$$\begin{aligned}
&T_0^{-1}\langle T(x, t)\rangle \\
&\quad = \exp\left(tc_1\frac{\partial^2}{\partial x^2}\right)\exp\left(tc_2\frac{\partial^4}{\partial x^4}\right)\delta(x - x_0) \\
&\quad = \exp\left(tc_2\frac{\partial^4}{\partial x^4}\right)(4\pi c_1 t)^{-1/2}\exp -\frac{(x - x_0)^2}{4c_1 t} \\
&\quad = \sum_{n=0}^{\infty}\frac{(tc_2)^n}{n!}\frac{\partial^{4n}}{\partial x^{4n}}(4\pi c_1 t)^{-1/2}\exp -\frac{(x - x_0)^2}{4c_1 t} \\
&\quad = \sum_{n=0}^{\infty}\frac{(tc_2)^n}{n!}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\delta(4n - m_1 - 2m_2)\frac{(4n)!}{m_1!\,m_2!\,2^{m_2}} \\
&\qquad \times\left\{-\frac{(x - x_0)}{2c_1 t}\right\}^{m_1}(-2c_1 t)^{-m_2}(4\pi c_1 t)^{-1/2}\exp -\frac{(x - x_0)^2}{4c_1 t}
\end{aligned} \tag{21}$$

where the last inequality follows directly from an application of DiBruno's formula.[3]

Note that the physical dimensions for c_1 and c_2 are (length)^2 (time)$^{-1}$ and (length)^4 (time)$^{-1}$, respectively. Consequently, a typical term in Eq. (21) has the summand

$$(c_2 t)^n \left(-\frac{(x - x_0)}{2c_1 t}\right)^{m_1} (-2c_1 t)^{-m_2}$$

which has physical dimensions of (length)^{4n} (length)$^{-m_1}$ (length)$^{-2m_2}$. The Dirac delta function $\delta(4n - m_1 - 2m_2)$ in Eq. (21) shows that all these "lengths" cancel out, and m_1 is always an even integer. The time dependence of this typical summand is clearly $(t)^{n-m_1-m_2}$, which for $n \geqslant 1$ always has a negative exponent. Thus, for asymptotically long times, only the $n = 0$ term in Eq. (21) persists, and this is merely the Gaussian term

$$(4\pi c_1 t)^{-1/2} \exp -\frac{(x - x_0)^2}{4c_1 t}$$

4. SPATIAL MOMENTS OF AVERAGE TEMPERATURE PROFILE

The average temperature profile $\langle T(x, t)\rangle$ is a nonnegative function and its integral over $(-\infty, \infty)$ is finite; consequently $\langle T(x, t)\rangle$ can be interpreted as a probability density function. The lower order spatial moments

$$\langle (x - x_0)^l \rangle = \int_{-\infty}^{\infty} (x - x_0)^l \langle T(x, t)\rangle \, dx \tag{22}$$

can be evaluated and shed light on the global behavior of $\langle T(x, t)\rangle$. When l is an odd integer, we have $\langle (x - x_0)^l\rangle = 0$ since $\langle T(x, t)\rangle$ is symmetric about x_0.

Perhaps the easiest way to obtain these moments is to multiply Eq. (18) by $(x - x_0)^l$ and integrate over x; the final result is

$$\begin{aligned} d/dt \langle (x - x_0)^l\rangle = {} & l(l - 1)c_1 \langle (x - x_0)^{l-2}\rangle \\ & + l(l - 1)(l - 2)(l - 3)c_2 \langle (x - x_0)^{l-4}\rangle \end{aligned} \tag{23}$$

For $l = 2$, this becomes

$$d/dt \langle (x - x_0)^2\rangle = 2c_1 \tag{24}$$

which has the solution

$$\langle (x - x_0)^2\rangle = 2c_1 t \tag{25}$$

since the initial condition is $T(x, 0) = \delta(x - x_0)$. This is the moment we would obtain if $\langle T(x, t)\rangle$ were in the form of a Gaussian.

For the fourth moment ($l = 4$), Eq. (23) becomes

$$d/dt \langle (x - x_0)^4 \rangle = 12c_1 \langle (x - x_0)^2 \rangle + 24c_2 \tag{26}$$

which has the solution

$$\langle (x - x_0)^4 \rangle = 12c_1^2 t^2 + 24c_2 t \tag{27}$$

The first term on the right-hand side is the fourth moment if $\langle T(x, T) \rangle$ were Gaussian, and is the dominant term when t is large.

In fact, all the moments tend to the Gaussian limiting values as t is made large. However, the Gaussian is precisely the form of the temperature pulse for the corresponding deterministic situation.[4] Thus the stochastic pulse shape tends to the corresponding deterministic pulse shape as time increases.

It is also possible to obtain the results listed in Eqs. (25) and (27) by direct integration of Eq. (21), although the analysis is tedious.

5. EQUATION FOR $\langle S(x, t)S(y, t) \rangle$

An equation for the covariance $\langle S(x, t)S(y, t) \rangle$ can be obtained by using the following procedure. First discretize the x axis by using n points $x_1, \ldots, x_n$. The following abbreviations are employed:

$$S_i \equiv S(x_i, t), \qquad \tilde{\mu}_i(t) \equiv \tilde{\mu}(x_i, t)$$
$$\langle \tilde{\mu}_i(t) \tilde{\mu}_j(t) \rangle \equiv 2R_{ij}\, \delta(t - s) \tag{28}$$

Also,

$$\partial^2/\partial x^2 \approx S_{ij} = A_{ik}A_{kj} = \delta_{ij} - 2\delta_{i-1,j} + \delta_{i-2,j} \tag{29}$$

where

$$A_{ik} = \delta_{ik} - \delta_{i-1,k} \tag{30}$$

Furthermore, we consider $f(S_1, \ldots, S_n; t)$ as the probability density function for the values taken on by $S_1, \ldots, S_n$.

The "continuity" equation for f is

$$\frac{\partial}{\partial t} f(S_1, \ldots, S_n; t) = -\frac{\partial}{\partial S_i} [\dot{S}_i f(S_1, \ldots, S_n; t)] \tag{31}$$

and from Eqs. (1) and (4), it follows that

$$\dot{S}_i = \nu_0[1 + \tilde{\mu}_i(t)] S_{ij} S_j \tag{32}$$

Consequently

$$\frac{\partial}{\partial t} f(S_1, \ldots, S_n; t) = -\frac{\partial}{\partial S_i} \{\nu_0[1 + \tilde{\mu}_i(t)] S_{ij} S_j f(S_1, \ldots, S_n; t)\} \tag{33}$$

which is a multiplicative stochastic process in the terminology of Fox.[2] Because the covariance function in Eq. (6) involves a delta function in time, only the first two cumulants are needed to obtain the average of Eq. (33), which works out to be

$$\begin{aligned}\frac{\partial}{\partial t}\langle f\rangle &= -\nu_0 \frac{\partial}{\partial S_i}(S_{ij}S_j\langle f\rangle) + \nu_0{}^2 \frac{\partial}{\partial S_i}\left(S_{ij}S_jR_{ii'}\frac{\partial}{\partial S_{i'}}S_{i'j'}S_{j'}\langle f\rangle\right)\\ &= -\nu_0 \frac{\partial}{\partial S_i}(S_{ij}S_j\langle f\rangle) - \nu_0{}^2 \frac{\partial}{\partial S_i}(S_{ij}R_{ij}S_{jj'}S_{j'}\langle f\rangle)\\ &\quad + \nu_0{}^2 \frac{\partial^2}{\partial S_i \partial S_{i'}}(S_{ij}S_jR_{ii'}S_{i'j'}S_{j'}\langle f\rangle) \end{aligned} \tag{34}$$

It follows that

$$\langle \dot{S}_i\rangle = \nu_0 S_{ij}\langle S_j\rangle + \nu_0{}^2 S_{ij}R_{ij}S_{jj'}\langle S_{j'}\rangle \tag{35}$$

This is a discretized version of

$$\frac{\partial}{\partial t}\langle S(x,t)\rangle = \nu_0 \frac{\partial^2}{\partial x^2}\langle S(x,t)\rangle + \nu_0{}^2 \frac{\partial^2}{\partial y^2} R(|x-y|)_{y=x}\frac{\partial^2}{\partial x^2}\langle S(x,t)\rangle \tag{36}$$

which agrees with Eq. (18) for the mean of $S(x, t)$.

In a similar fashion, it can be shown that

$$\begin{aligned}&(d/dt)\langle S_l(t)S_k(t)\rangle\\ &\quad = \nu_0 S_{lj}\langle S_j(t)S_k(t)\rangle + \nu_0 S_{kj}\langle S_j(t)S_l(t)\rangle\\ &\qquad + \nu_0{}^2 S_{lj}R_{lj}S_{jj'}\langle S_{j'}(t)S_k(t)\rangle + \nu_0{}^2 S_{kj}R_{kj}S_{jj'}\langle S_{j'}(t)S_l(t)\rangle\\ &\qquad + 2\nu_0{}^2 S_{lj}R_{lk}S_{kj'}\langle S_j(t)S_{j'}(t)\rangle\end{aligned} \tag{37}$$

This is a discretized version of the equation for the covariance

$$\begin{aligned}&\frac{\partial}{\partial t}\langle S(xt)S(yt)\rangle\\ &\quad = \nu_0\frac{\partial^2}{\partial x^2}\langle S(x,t)S(y,t)\rangle + \nu_0\frac{\partial^2}{\partial y^2}\langle S(x,t)S(y,t)\rangle\\ &\qquad + \nu_0{}^2\frac{\partial^2}{\partial z^2}R(|x-z|)\Big|_{z=x}\frac{\partial^2}{\partial x^2}\langle S(x,t)S(y,t)\rangle\\ &\qquad + \nu_0{}^2\frac{\partial^2}{\partial z^2}R(|y-z|)\Big|_{y=z}\frac{\partial^2}{\partial y^2}\langle S(x,t)S(y,t)\rangle\\ &\qquad + 2\nu_0{}^2 R(|x-y|)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\langle S(x,t)S(y,t)\rangle\end{aligned} \tag{38}$$

A somewhat simpler version can be obtained; define

$$\sigma(x, y, t) \equiv \langle S(x, t)S(y, t)\rangle - \langle S(x, t)\rangle\langle S(y, t)\rangle \tag{39}$$

Upon using Eqs. (36), (38), and (15), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\sigma(x, y, t) \\
&= \nu_0\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma(x, y, t) \\
&\quad + \left[\nu_0{}^2 R''(0)\frac{\partial^2}{\partial x^2} + \nu_0{}^2 R''(0)\frac{\partial^2}{\partial y^2}\right]\sigma(x, y, t) \\
&\quad + 2\nu_0{}^2 R(|x - y|)\frac{\partial^4}{\partial x^2\,\partial y^2}[\sigma(x, y, t) + \langle S(x, t)\rangle\langle S(y, t)\rangle]
\end{aligned} \tag{40}$$

We have not made any attempt to solve this equation.

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